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ANALYSIS FOR CONCENTRIC-DOUBLE INCLUSIONS DISPERSED IN CONTINUOUS MEDIA

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Abstract—The behaviors of concentric-double inclusions dispersed in continuous media are investigated theoretically to find some possibilities of improving toughness of composite materials by dispersing double-inc'usions instead of single-inclusions. The general solutions of the Stokes equation, expressed in terms of the spherical harmonics, are used for analyzing the problems that are related to the concentric-double inclusions. From the analysis, it is found that the pressure and stress fields inside and outside the inclusion can be modified by changing the modulus ratios and the thickness of shell layer. Especially, the positions of the minimum pressure points and the maximum stress points turn out to be controllable with some degree of freedom.

Key words: *Concentric-double Elastic Inclusion, Stokes Equation, Toughness of Composite Material*

INTRODUCTION

Multiphase systems that consist of dispersed particles in continuous media are widely found both in nature and in man-made situations. The words closely related to multiphase systems are suspensions, slurries, colloids, composite materials, etc. The systems of particles suspended in continuous media generally exhibit a variety of remarkable macroscopic properties. In practical applications, we are interested in such collective effects of particles that can be represented as the effective transport properties such as viscosity, modulus, mobility, conductivity, etc. Thus, the main efforts of developing theories of multiphase systems have been focused on the predictability of macroscopic properties by considering the behaviors and characteristics of individual particles.

Lately active experimental investigations have been performed to obtain some excellent macroscopic properties of the composite materials [Fowler et al., 1987; Gebizlioglu et al., 1990; Lovell et al., 1991; Laurienzo et al., 1992]. It is common that multiphase structures are formed due to lack of miscibility in a molecular level when the polymers of different properties are mixed. In some cases, unique and excellent macroscopic properties can be obtained by taking advantage of such structures formed by phase separation. Thus, various investigators have sought the ways of improving toughness of material by adjusting the pressure and stress fields when the material is subject to certain strain fields. One way of achieving such modification is to make composite systems of different phases with different properties via phase separation or solidifying the multiphase fluids. Most experimental works thus far, however, have relied on trial-and-error method. In other words, not many experiments have been performed under *a priori* theoretical guidances because of lack of theoretical results in this fields [Matonis, 1969; Ricco et al., 1980; Moshev, 1980].

In the present work, as an elementary step toward the development of theories for toughness improvement of composite materials, we have considered a concentric-double elastic inclusion under the given strain fields. As a solution method, we have used the general solution of the Stokes equation, expressed in terms of the spherical harmonics. With the analytical solutions, we have analyzed the pressure-stress fields inside and outside the inclusion to find some possibility of improving toughness while the macroscopic bulk properties of the composite material remain unchanged. As will be shown later, we have found the possibility of changing the positions of failure points by modifying the modulus ratios and the thickness of shell structure. Although very simple in analysis, the results are certainly expected to play a role as a theoretical guidance for the experimental works for improving toughness of composite materials.

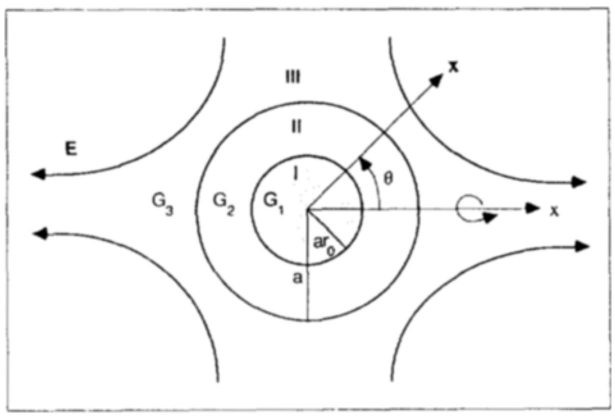

Fig. 1. A concentric-double inclusion in a strain field (The strain field is axisymmetric with respect to x-axis).

PROBLEM STATEMENT AND DEVELOPMENT

We consider a concentric-double-inclusion in a continuous elastic medium subject to a given strain field as sketched in Fig. 1. The concentric-double-inclusion consists of the inner part

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(phase I) of modulus G_1 and of radius ar_0 ($r_0 \leq 1$), and the shell part (phase II) of modulus G_2 and of thickness $a(1-r_0)$. The inclusion is dispersed in the continuous elastic medium (phase III). The far field strain tensor (or undisturbed strain tensor field) is denoted by **E**. For simplicity, we assume that the whole system is under isothermal condition and that each phase is incompressible. It is also assumed that there is no interfacial energy between phases. Thus, the traction ($\mathbf{n}\cdot\mathbf{T}$) must be continuous across the phase boundaries.

The problem we want to solve is to find the pressure and stress fields in each phase for a given **E** field. For the given far field strain tensor, the phase boundaries undergo deformation and in general the problem becomes extremely difficult because the shapes of the phase boundaries must be determined simultaneously. Thus, for simplicity, as in many other stress analysis problems of solid mechanics, the modulus of each phase is assumed to be sufficiently large so that the deformations of phase boundaries are negligibly small (In other words, $\|\mathbf{E}\|$ is assumed to be sufficiently small even though $\|G\mathbf{E}\|$ has some finite value.). Under this assumption, we can safely neglect the effect of deformations of phase boundaries and we can apply the boundary conditions at $r=a$ and $r=ar_0$. The problem will be solved via spherical harmonics to obtain the pressure and stress fields inside and outside the concentric-double-inclusion. Then the solution will be used to estimate the macroscopic modulus of the composite material with concentric-double-inclusions in the dilute regime.

Under the assumptions stated above, the governing equations for the phases, I, II and III, are given by the Stokes equation and the continuity equation (due to incompressibility assumption)

$$-\tilde{\nabla}\tilde{p}_{(n)}+G_{(n)}\tilde{\nabla}^2\tilde{\mathbf{u}}_{(n)}=0, \tag{1}$$

$$\tilde{\nabla}\cdot\tilde{\mathbf{u}}_{(n)}=0,\ (n=1,\ 2,\ 3) \tag{2}$$

where $\tilde{p}_{(n)}$ is the hydrostatic pressure of the n-th phase, $\tilde{\mathbf{u}}_{(n)}$ the displacement vector of the n-th phase, and $G_{(n)}$ the shear modulus (Hereinafter, the mathematical accent tilde is for the *dimensional* variables of pressure, displacement vector, and stress and strain tensors.). As shown in Eqs. (1) and (2), the governing equations are the same for the viscous fluid flows with the velocity vector **u** and the viscosity μ. Boundary conditions at the interfaces are given by

$$\tilde{\mathbf{u}}_{(1)}=\tilde{\mathbf{u}}_{(2)}$$
$$\mathbf{n}\cdot\tilde{\mathbf{T}}_{(1)}=\mathbf{n}\cdot\tilde{\mathbf{T}}_{(2)}\ \text{at}\ |\tilde{\mathbf{x}}|=ar_0, \tag{3}$$

$$\tilde{\mathbf{u}}_{(2)}=\tilde{\mathbf{u}}_{(3)}$$
$$\mathbf{n}\cdot\tilde{\mathbf{T}}_{(2)}=\mathbf{n}\cdot\tilde{\mathbf{T}}_{(3)}\ \text{at}\ |\tilde{\mathbf{x}}|=a, \tag{4}$$

where $\tilde{\mathbf{T}}$ is the stress tensor defined by

$$\tilde{\mathbf{T}}=-\tilde{p}\mathbf{I}+G(\tilde{\nabla}\tilde{\mathbf{u}}+\tilde{\nabla}\tilde{\mathbf{u}}^T). \tag{5}$$

The far field strain tensor is given by

$$\tilde{\mathbf{E}}=E_c\begin{pmatrix}1 & 0 & 0\\ 0 & -\frac{1}{2} & 0\\ 0 & 0 & -\frac{1}{2}\end{pmatrix}.$$

In order to nondimensionalize the governing equations and boundary conditions, we introduce the following characteristic scales:

$$l_c=a,\ u_c=E_c l_c,\ p_c=\frac{G_3 u_c}{l_c}. \tag{6}$$

Then the governing equations appropriate to the problem described above are given by

$$\text{Phase I}: \nabla\cdot\mathbf{u}_{(1)}=0,\ \nabla p_{(1)}=\alpha\nabla^2\mathbf{u}_{(1)}, \tag{7}$$

$$\text{Phase II}: \nabla\cdot\mathbf{u}_{(2)}=0,\ \nabla p_{(2)}=\beta\nabla^2\mathbf{u}_{(2)}, \tag{8}$$

$$\text{Phase III}: \nabla\cdot\mathbf{u}_{(3)}=0,\ \nabla p_{(3)}=\nabla^2\mathbf{u}_{(3)}, \tag{9}$$

where α and β are the parameters defined as

$$\alpha=\frac{G_1}{G_3},\ \beta=\frac{G_2}{G_3}. \tag{10}$$

The suitable boundary conditions are

$$\mathbf{u}_{(1)}\rightarrow\text{finite as}\ |\mathbf{x}|\rightarrow 0, \tag{11}$$

$$\mathbf{u}_{(1)}=\mathbf{u}_{(2)}$$
$$\mathbf{n}\cdot\mathbf{T}_{(1)}=\mathbf{n}\cdot\mathbf{T}_{(2)}\ \text{at}\ |\mathbf{x}|=r_0, \tag{12}$$

$$\mathbf{u}_{(2)}=\mathbf{u}_{(3)}$$
$$\mathbf{n}\cdot\mathbf{T}_{(2)}=\mathbf{n}\cdot\mathbf{T}_{(3)}\ \text{at}\ |\mathbf{x}|=1, \tag{13}$$

$$\mathbf{u}_{(3)}\rightarrow\mathbf{E}\cdot\mathbf{x}\ \text{as}\ |\mathbf{x}|\rightarrow\infty. \tag{14}$$

From the above dimensionless formulation, it should be noted that the global problem has three parameters α, β, and r_0, which play important roles in the discussions of the results.

The solution method of the Stokes equation via the spherical harmonics is well explained in the reference [Leal, 1992]. But here we briefly review the method to explain our solutions for the concentric-double inclusion problem. The Stokes equation with the continuity equation

$$\mu\nabla^2\mathbf{u}=\nabla p,\ \nabla\cdot\mathbf{u}=0$$

is linear and has some nice properties. The first is that the pressure is harmonic, i.e.

$$\nabla^2 p=0. \tag{15}$$

The second feature is that a general solution of the Stokes equation is given by

$$\mathbf{u}=\frac{\mathbf{x}}{2\mu}p+\mathbf{u}^{(H)}, \tag{16}$$

where $\mathbf{u}^{(H)}$ is the homogeneous solution. Substituting Eq.(16) into the Stokes equation, we can easily see that $\mathbf{u}^{(H)}$ is also harmonic, i.e.

$$\nabla^2\mathbf{u}^{(H)}=0. \tag{17}$$

In order to satisfy the continuity equation, the harmonic function $\mathbf{u}^{(H)}$ must satisfy the condition,

$$\nabla\cdot\mathbf{u}^{(H)}=-\frac{1}{2\mu}\left[3p+\mathbf{x}\cdot\nabla p\right]. \tag{18}$$

To develop a general solution for the Stokes equation, we introduce harmonic functions, expressed in a coordinate-independent vector form using the position vector **x**. Harmonic functions, the fundamental solutions of the Laplace equation, involve only the general position vector **x** and its magnitude $r=|\mathbf{x}|$. It is convenient to divide the harmonic functions into two categories : *decaying harmonics* and *growing harmonics*. The decaying harmonics whose magnitude decrease with $|\mathbf{x}|$ are represented by means of higher-order gradients of 1/r. Some decaying harmonics are

$$\frac{1}{r},\ \frac{\mathbf{x}}{r^3},\ \frac{\mathbf{xx}}{r^5}-\frac{\mathbf{I}}{3r^3},\ \text{etc.} \tag{19}$$

On the other hand, some of the growing harmonics whose magnitude increases with $|\mathbf{x}|$ are

$$1,\ \mathbf{x},\ \mathbf{xx}-\frac{r^2}{3}\mathbf{I},\ \text{etc.} \tag{20}$$

Now, the general solution for $\mathbf{u}$ and p in our concentric-double inclusion problem can be constructed in terms of the strain tensor $\mathbf{E}$ and the harmonic functions in vector form. We begin with the pressure $p_{(3)}$ and the velocity $\mathbf{u}_{(3)}$ for the region III. In this case, as we can see in Eq. (9), we may put $\mu=1$ for the general solution procedure. For simplicity, we reformulate the problem in terms of the disturbance variables $\mathbf{u}'_{(3)}$ and $p'_{(3)}$ defined as

$$\mathbf{u}'_{(3)}=\mathbf{u}_{(3)}-\mathbf{E}\cdot\mathbf{x}. \tag{21}$$

Then the boundary conditions are changed to

$$\mathbf{u}'_{(3)}=\mathbf{u}_{(2)}-\mathbf{E}\cdot\mathbf{x}\ \text{at}\ |\mathbf{x}|=1, \tag{22}$$

$$\mathbf{u}'_{(3)}\to 0\ \text{as}\ |\mathbf{x}|\to\infty. \tag{23}$$

Thus, $\mathbf{u}'_{(3)}$, $p'_{(3)}$ are decaying functions of r. To construct a solution for $\mathbf{u}'_{(3)}$ and $p'_{(3)}$, we begin with the pressure $p'_{(3)}$ which is a harmonic scalar function. Since p'_3 is a decaying harmonic function that is linear in $\mathbf{E}$, the only true scalar that can be formed by $\mathbf{E}$ and the decaying harmonic functions in Eq. (19) is

$$p'_{(3)}=C_1\mathbf{E}:\left[\frac{\mathbf{xx}}{r^5}-\frac{\mathbf{I}}{3r^3}\right]. \tag{24}$$

Since trace of $\mathbf{E}$ equals zero due to the incompressibility assumption, this simplifies to

$$p'_{(3)}=C_1\left[\frac{\mathbf{x}\cdot\mathbf{E}\cdot\mathbf{x}}{r^5}\right]. \tag{25}$$

The constant C_1 will be determined by applying the boundary conditions for $\mathbf{u}'_{(3)}$. On the other hand, $\mathbf{u}^{(H)}_{(3)}$ is a decaying harmonic function that is linear in $\mathbf{E}$ and is a true vector. The combination of $\mathbf{E}$ and the vector harmonic functions which satisfies these conditions is obtained by

$$\mathbf{u}^{(H)}_{(3)}=C_2\left[\frac{\mathbf{E}\cdot\mathbf{x}}{r^3}\right]+C_3\left[\frac{(\mathbf{x}\cdot\mathbf{E}\cdot\mathbf{x})\mathbf{x}}{r^7}\right]-\frac{2}{5}C_3\left[\frac{\mathbf{E}\cdot\mathbf{x}}{r^5}\right]. \tag{26}$$

The constants C_2 and C_3 are arbitrary, apart from the constraint Eq. (18), which insures that the continuity equation is satisfied. The condition of Eq. (18) yields $C_2=0$. Thus, combining Eqs. (25) and (26) in terms of the general solution form, Eq. (16), we find that the velocity field $\mathbf{u}_{(3)}$ is given by

$$\mathbf{u}_{(3)}=\frac{C_1}{2}\left[\frac{(\mathbf{x}\cdot\mathbf{E}\cdot\mathbf{x})\mathbf{x}}{r^5}\right]+C_3\left[\frac{(\mathbf{x}\cdot\mathbf{E}\cdot\mathbf{x})\mathbf{x}}{r^7}\right]-\frac{2}{5}C_3\left[\frac{\mathbf{E}\cdot\mathbf{x}}{r^5}\right]+(\mathbf{E}\cdot\mathbf{x}). \tag{27}$$

By following the same procedures, we can obtain $\mathbf{u}_{(1)}$, $p_{(1)}$, $\mathbf{u}_{(2)}$ and $p_{(2)}$ in terms of growing harmonics, and decaying and growing harmonics, respectively.

$$p_{(1)}=C_{10}(\mathbf{x}\cdot\mathbf{E}\cdot\mathbf{x}), \tag{28}$$

$$p_{(2)}=C_4\left[\frac{\mathbf{x}\cdot\mathbf{E}\cdot\mathbf{x}}{r^5}\right]+C_5(\mathbf{x}\cdot\mathbf{E}\cdot\mathbf{x}), \tag{29}$$

$$\mathbf{u}_{(1)}=-\frac{2}{21\alpha}C_{10}(\mathbf{x}\cdot\mathbf{E}\cdot\mathbf{x})\mathbf{x}+\frac{5r^2}{21\alpha}C_{10}+C_{11}(\mathbf{E}\cdot\mathbf{x}). \tag{30}$$

$$\mathbf{u}_{(2)}=\frac{C_4}{2\beta}\left[\frac{(\mathbf{x}\cdot\mathbf{E}\cdot\mathbf{x})\mathbf{x}}{r^5}\right]-\frac{2}{21\beta}C_5(\mathbf{x}\cdot\mathbf{E}\cdot\mathbf{x})\mathbf{x}+C_7\left[\frac{(\mathbf{x}\cdot\mathbf{E}\cdot\mathbf{x})\mathbf{x}}{r^7}\right]-\frac{2}{5}C_7\left[\frac{\mathbf{E}\cdot\mathbf{x}}{r^5}\right]+\left(\frac{5r^2}{21\beta}C_5+C_8\right)(\mathbf{E}\cdot\mathbf{x}), \tag{31}$$

To determine the eight constants C_1, C_3, C_4, C_5, C_7, C_8, C_{10}, C_{11}, and thus to complete our solution of the problem, we apply the boundary conditions (12) and (13) at r=1 and r_0. Then, we can obtain the eight equations that have 8 unknown constants. The eight algebraic equations may be solved to calculate the eight constants for a given set of parameters α, β, and r_0.

RESULTS AND DISCUSSION

1. Pressure and T_{xx} Fields

Among the major concerns in the processing of composite materials are the occurrence of material failure and the propagation of fractures when the material is subject to certain strain fields. Two quantities that are closely related to the failure points and the propagation of fractures are the pressure and T_{xx} (when E_{xx} is the major component of the strain). As well known, the pressure field in a single homogeneous medium is harmonic. Therefore the minimum and maximum values should exist only on the boundary. Thus, in the case of infinite homogeneous medium, the pressure field should be constant since there is no boundary. Similarly, the concentration of stress does not occur in a homogeneous medium when the material undergoes strain fields. However, for a composite material, that is not the case. Due to the interactions of the continuous and dispersed phases, the pressure and stress cannot be uniform and there exist maximum stress points and minimum pressure points. We call this phenomenon the stress concentration in the composite materials. From a standpoint of material failure, the minimum pressure point corresponds to the point where cavitation occurs. On the other hand, T_{xx}, which is the x-directional force acting on the unit area of the plane whose normal is in x-direction, can provide a measure of possibility of fracture propagation. Therefore, if the pressure minimum point coincides with the maximum T_{xx} point, then it is very probable that the material failure occurs at the point.

As mentioned earlier, the main objective of the present work is to find some possibilities of adjusting the pressure-stress fields by modifying the parameters α, β, and r_0. Now, let us start with our discussion for the cases of single inclusions.

1-1. Single Inclusions

In this subsection, we are concerned with the pressure-stress response of single spherical inclusion (α=β case) when the material is subject to given strains. Here, we consider two cases. One is for α=β<1 and the other is for α=β>1. To investigate the effect of modulus of inclusion, we have analyzed the pressure and stress fields (T_{xx}). In Fig. 2, the pressure and stress fields are shown for a system with a single inclusion. The stress field is for the component T_{xx}, which is the stress in x-direction acted on x-plane. When α=β<1, i.e., the modulus of inclusion is less than that of matrix [Fig. 2(a)], the minimum value of pressure and the maximum value of stress are located at the same points of the inclusion boundary, which are the equator points (θ=π/2) with respect to the direction of given strains. On the other hand,

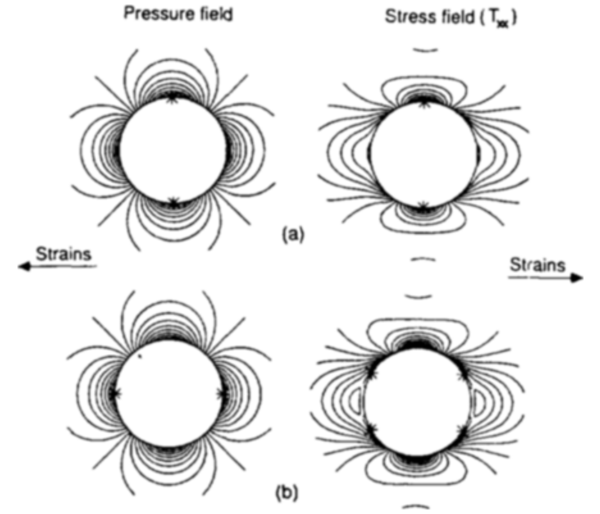

Fig. 2. Pressure and stress fields for a system with a single inclusion when (a) $\alpha=\beta=0.5<1$, (b) $\alpha=\beta=2.0>1$.

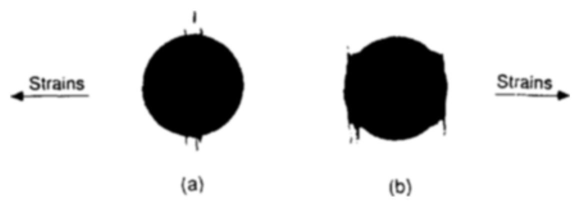

Fig. 3. Photographs of crazed sampled (reprinted from Wang et al., 1971).
(a) rubber ball, (b) steel ball

in the case of harder inclusion than the matrix, the pressure minimum points are located at the poles. But the stress maximum points are located between the poles and the equator.

In both cases, we may assume that the cavitation occurs at the minimum pressure points and the fracture is propagated from the stress maximum point to the matrix phase when the strains are imposed. From the pressure and T_{xx} fields in Fig. 2, we can expect that fracture occurs at the equator ($\theta=\pi/2$) in the case of $\alpha=\beta<1$ but fracture may occur in the wider region between the pole and the maximum T_{xx} point in the case of $\alpha=\beta>1$. The above point may well be verified by the experimental results of Wang et al. [1971]. Photograph in Fig. 3(a) shows how the fracture propagates into the matrix phase in the $\alpha=\beta<1$ case. As shown in the photograph, propagation starts at the equator of the rubber ball. As shown in Fig. 3(b), our analytical prediction for the $\alpha=\beta>1$ case is also consistent with the experimental result on the case of steel ball. The numerical values of the minimum pressure and the maximum stress are tabulated for the $\alpha=\beta<1$ cases and $\alpha=\beta>1$ cases in Tables 1 and 2, respectively. As we can see in the tables, the degree of stress concentration or pressure minimization increases as the relative difference of moduli of inclusion and matrix increases. In other words, the degree of stress concentration increases as the degree of non-homogeneity of the material increases.

Another conclusion we may draw from the analysis for the single inclusion case is that microscopic failure is practically inevitable because of the high stress concentration for either case of

Table 1. Numerical values of pressure minimum and stress maximum for the case of elastic single inclusion with $\alpha=\beta<1$

$\alpha=\beta$	T_{max}	P_{min}
0.01	6.589	−3.278
0.1	5.937	−2.813
0.5	3.750	−1.250
0.9	2.292	−0.208

Table 2. Numerical values of pressure minimum and stress maximum for the case of elastic single inclusion with $\alpha=\beta>1$

$\alpha=\beta$	T_{max}	P_{min}
1.5	2.500	−0.833
2.0	3.211	−1.428
10.0	5.318	−3.913
100.0	7.214	−4.877

$\alpha=\beta<1$ or $\alpha=\beta>1$. This point provides us a strong motivation to seek other ways of modifying stress fields. As one of such efforts, we consider concentric-double inclusions, which will be discussed next.

1-2. Concentric-Double-Inclusions

As mentioned earlier, we are concerned with the effects of the first and second layers with the modulus ratios α and β on the stress and pressure fields in the composite materials. The thickness of the second layer, $1-r_0$, is also an important factor in this analysis.

Firstly, we have considered the double inclusions of the cases of gradual softening ($\alpha<\beta<1$) and gradual hardening ($\alpha>\beta>1$) as shown in Figs. 4(a) and 4(b). To see the effect of thickness of the second layer, we have considered $r_0=0.5$ and $r_0=0.95$ for both cases. When the thickness of the second layer is 0.05 (i.e., $r_0=0.95$), the behaviors of double inclusions are more or less the same as those of single inclusions. However, when we have reduced r_0 to 0.5 (increased the thickness of the second layer), remarkable changes in the behavior have been observed. As we can see in Figs. 4(a) and (b), the maximum stress points moved to the inner boundary (Compare with the cases of $r_0=0.95$). Especially, in the case of gradual softening ($\alpha<\beta<1$), the pressure minimum points have also moved to the inner boundary [Fig. 4(a)]. These changes in the response to the given strain fields have very significant implications in the studies of composite materials. As mentioned earlier, for composite materials with single inclusion it is inevitable that material failures would propagate into the matrix. However, in the cases of double inclusion composite material, the failure propagation may be confined in the second layer. Thus, we may improve the toughness of the composite material significantly in some cases.

Secondly, we have tried to modify the stress fields around the very hard inclusion by covering it with the second layers. For very hard center inclusion ($\alpha=10$, $r_0=0.5$), we have considered both very soft ($\beta=0.001$) and moderately hard ($\beta=1.5$) second layers. As shown in Fig. 5, the hard inclusion covered with very soft layer of thickness $r_0=0.5$ behaves just like a soft single inclusion as expected. However, the one with moderately soft layer behaves like the inclusion of gradual hardening ($\alpha>\beta>1$) and the pressure minimum points and the stress maximum points have moved to the inner boundary.

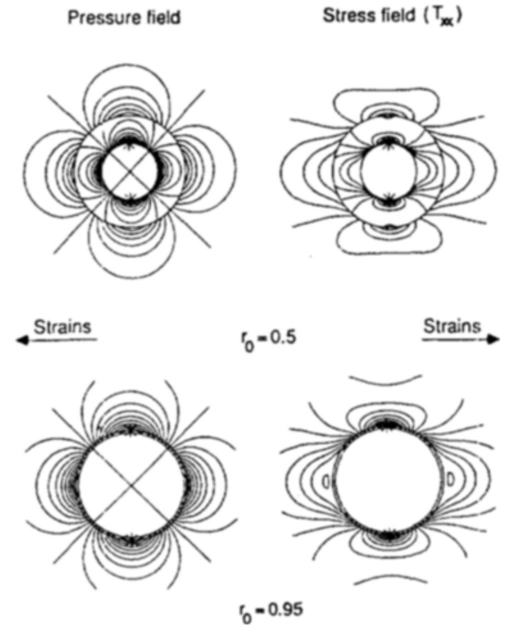

Fig. 4(a). Pressure and stress fields for double inclusions of gradual softening ($\alpha=0.5$, $\beta=0.8$) when r_0 values are 0.5 and 0.95.

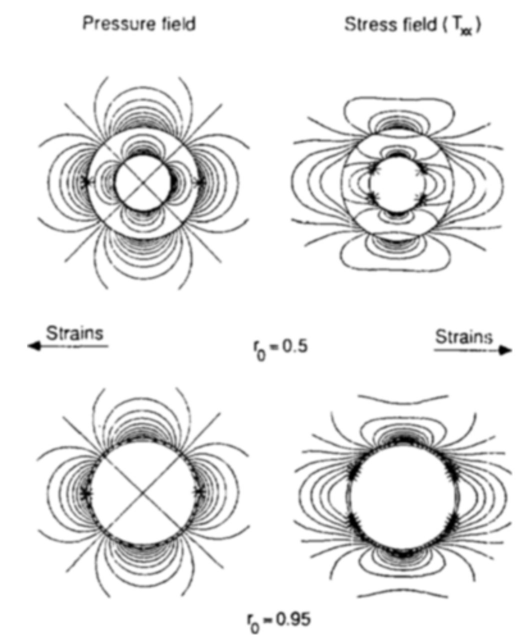

Fig. 4(b). Pressure and stress fields for double inclusions of gradual hardening ($\alpha=2.0$, $\beta=1.5$) when r_0 values are 0.5 and 0.95.

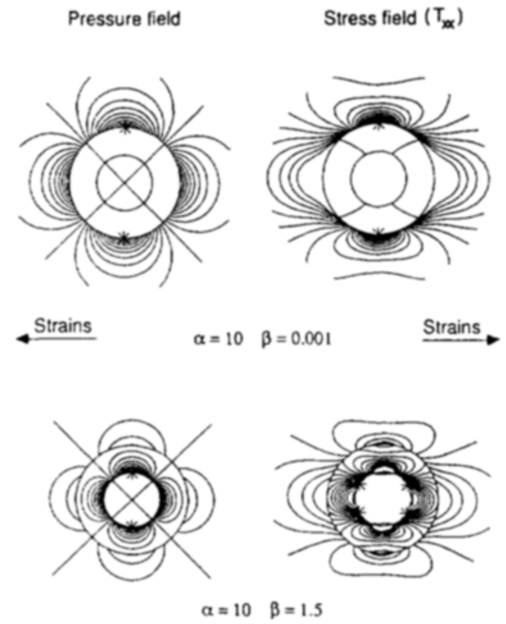

Fig. 5. Pressure and stress fields for inclusions of very hard cores ($\alpha=10$, $r_0=0.5$) covered with very soft ($\beta=0.001$) and moderately hard ($\beta=1.5$) material.

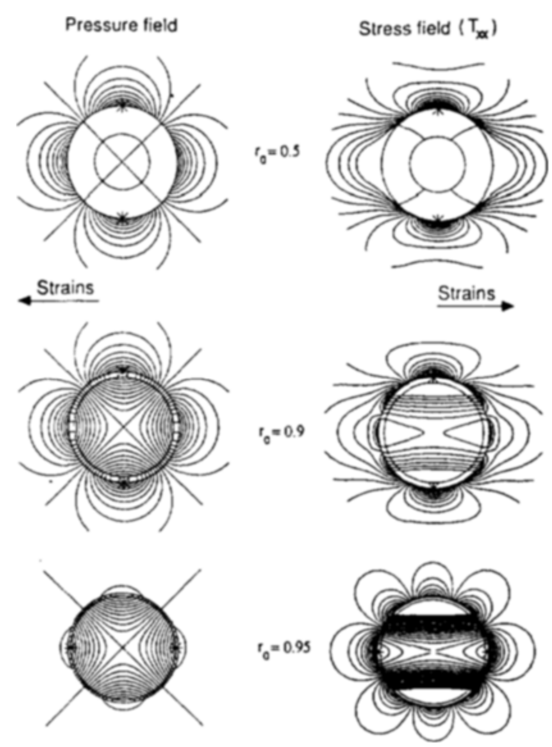

Fig. 6. Pressure and stress fields for inclusions of very hard cores ($\alpha=10$) covered with rubbery material ($\beta=0.001$) of various values of thickness ($r_0=0.5$, 0.9, 0.95).

In the third, we have investigated the effect of covering of hard core ($\alpha=10$) with rubbery material ($\beta=0.001$) for various covering thickness. The results are shown in Fig. 6. As shown in the figure, the double inclusions behave just like soft single inclusions dispersed in the matrix until r_0 value increases to $r_0=0.9$. However, fundamental change in the response has been observed when r_0 reaches 0.95, at which the pressure minimum occurs at the poles of the inner boundary, and the stress maximum occurs at the poles of the outer boundary. Since the probability that cavitation occurs at the pressure minimum point is very high, microscopic cavities may be formed near the inner boundary by coating the hard core with a very thin rubbery materials.

Finally, in Fig. 7, we have shown the results for the moderately hard core coated with very soft material. We have investigated two cases of medium hardness ($\alpha=0.8$ and $\alpha=1.5$). For both

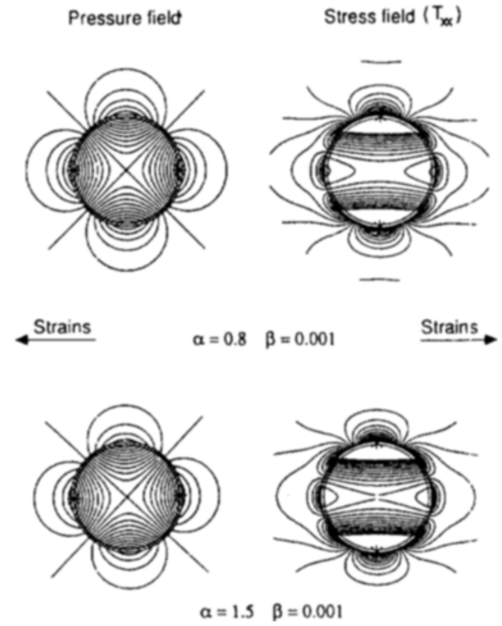

Fig. 7. Pressure and stress fields for inclusions of the cores of medium hardness ($\alpha=0.8$, $\alpha=1.5$) coated with very thin soft material ($\beta=0.001$, $r_0=0.95$).

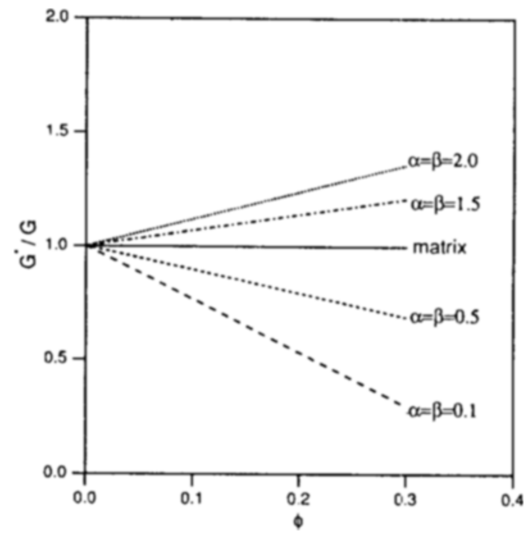

Fig. 8. Effects of the modulus of binded single inclusion ($\alpha=\beta$) on the relative modulus (G^*/G) in dilute regime.

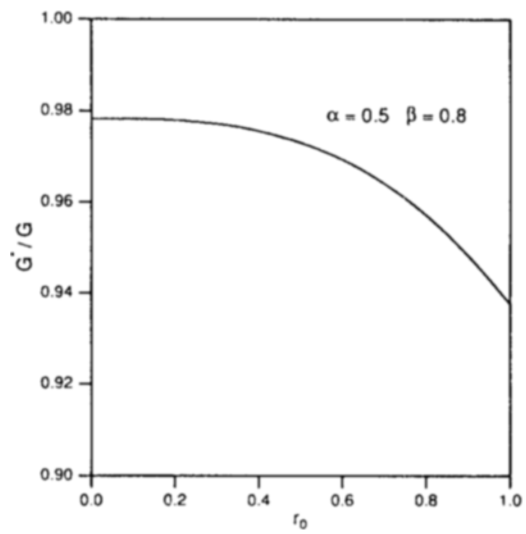

Fig. 9. Effects of r_0 on the relative modulus (G^*/G) of double inclusion system of gradual softening ($\alpha=0.5$, $\beta=0.8$) when ϕ is 0.1.

cases, we have found quite interesting behaviors. The pressure minimum point occurs at the poles of the inner boundary, while the maximum T_{xx} points occur at the equator of the outer boundary. Since two points are far apart, it is probable for microscopic cavities to be formed near the inner boundary without fracture propagation when the composite materials are subject to strain fields.

So far we have discussed the behaviors of the double inclusions under the given strain fields. Although the results on the double inclusions are limited, we have found the possibilities of modifying the characteristics of the stress and pressure fields, which cannot be achieved in the cases of single inclusions. Therefore, it is expected that the results from the analyses shown above may provide guidelines for designing composite materials with improved properties.

2. Effective Modulus of Concentric-Double-Inclusion System

The problem of deducing macroscopic rheological behavior of heterogeneous system from its microscopic structural properties has received considerable attention. In the present section, we want to predict the effective modulus of dilute composite materials which are composed of concentric-double-inclusions with arbitrary α, β, and r_0.

Batchelor [1970] proposed a relation between the bulk stress and the stress of individual particles of any concentrations. The relation is

$$\Sigma_{ij}-\delta_{ij}\Sigma_{kk}=2\mu E_{ij}+\Sigma_{ij}^{(P)}, \tag{32}$$

where the first term on the right-hand side is the stress that would be generated due to the ambient field in the absence of the particles. The second term on the right-hand side of Eq. (32) is the particle stress that is given by

$$\Sigma_{ij}^{(P)}=\frac{1}{V}\Sigma S_{ij} \tag{33}$$

where S_{ij} is the force dipole strength which depends on the size, shape and orientation of the particles. In the dilute case, the stresslet S_{ij} is appeared in the solution of the Stokes equation for the disturbance variable $\mathbf{u}'$ in the form

$$u'_k=-\frac{3}{2}\left(\frac{S_{ij}}{4\pi\mu}\right)\left[\frac{x_i x_j x_k}{r^5}\right]+\cdots. \tag{34}$$

Hence the first approximation to the particle stress [Batchelor and Green, 1972a, 1972b] in terms of S_{ij} is given as

$$\Sigma_{ij}^{(P)}=\frac{3\phi}{4a^3\pi}S_{ij}, \tag{35}$$

where ϕ is the volume fraction of the spherical particles of radius a. We have followed Batchelor and Green to obtain the following expression for the effective modulus G^*,

$$G^*=G(1+\gamma\phi), \tag{36}$$

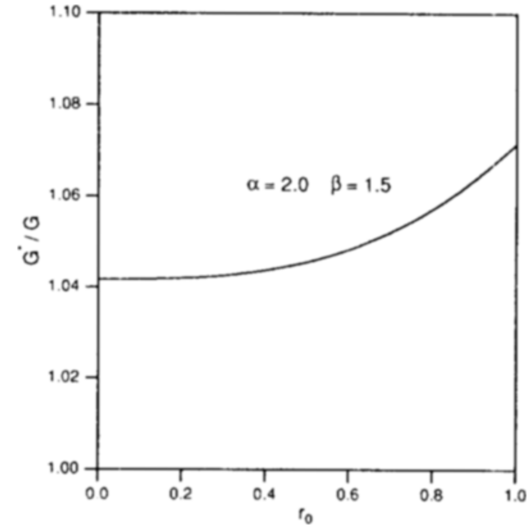

Fig. 10. Effects of r_0 on the relative modulus (G*/G) of double inclusion system of gradual hardening (α=2.0, β=1.5) when φ is 0.1.

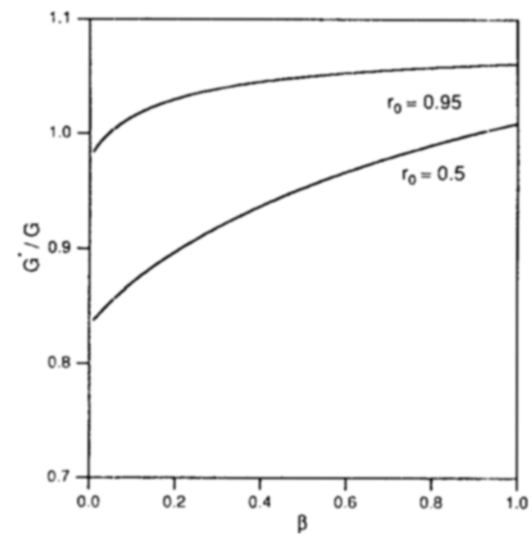

Fig. 11. Effects of the modulus of the secondary layer for r_0=0.5 and r_0=0.95 on the relative modulus G*/G when α and φ are 2.0 and 1.0, respectively.

where γ is defined as

$$\gamma \equiv -\frac{C_1}{2}. \tag{37}$$

In Eq. (37), C_1 is the constant that was introduced earlier in Eq. (24). It can be computed as a function of α, β, and r_0 by applying the boundary conditions as mentioned earlier.

The results obtained for various α, β, r_0 values are summarized in Figs. 8 to 11. In all figures, however, we must note that it is assumed that the concentration of the particle is small. As shown in Fig. 8, the relative modulus increases as the modulus of binded single particle (α=β) increases. The results shown in Fig. 8 are already well known from the theories of single inclusions. To investigate the effect of r_0, we have considered two cases as shown in Figs. 9 and 10. In Fig. 9, the relative modulus

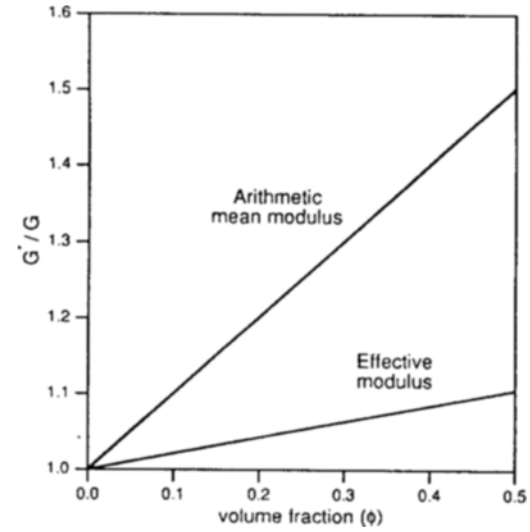

Fig. 12. Comparison of effective modulus with arithmetic mean modulus for dilute double inclusions (α=2.0, β=1.5).

decreases as r_0 increases when the volume fraction φ is 0.1. This is obvious because as r_0 increases the volume of the first layer with lower modulus increases. A similar observation can be made also in Fig. 10. In Fig. 11, the effects of the modulus of the second layer for fixed r_0=0.5 and 0.95 are shown. In Fig. 12, the effective modulus is compared with the arithmetic mean. The effective modulus is lower than the arithmetic mean value as shown by the general theorem of Batchelor [1974].

CONCLUSION

The behaviors of concentric-double-elastic inclusions dispersed in continuous media have been investigated theoretically. From the analyses for the pressure and stress fields, it has been found that the positions of minimum pressure points and the maximum stress points can be controlled with some degree of freedom by changing the modulus ratios and the thickness of the shell layer. Thus, it is expected that the results from the present work may provide good guidelines for designing composite materials with improved properties.

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NOMENCLATURE

a	: characteristic length scale
E	: far field strain tensor
E_{xx}	: x-directional strain component acting on the plane whose normal is in x-direction
$G_{(n)}$	: shear modulus of the nth phase
$p_{(n)}$	: pressure of the nth phase
p'	: disturbance pressure
r_0	: shell thickness parameter ($r_0 \leq 1$)
S_{ij}	: force dipole strength

T_{xx} : x-directional stress component acting on the plane whose normal is in x-direction
$\mathbf{u}_{(n)}$: displacement vector of the nth phase
$\mathbf{u}'$: disturbance displacement vector
$\mathbf{u}^{(H)}$: homogeneous solution of $\mathbf{u}$
V : volume of composite material
$\mathbf{x}$: position vector

Greek Letters

α : ratio of shear modulus G_1 to G_3
β : ratio of shear modulus G_2 to G_3
μ : viscosity of fluid medium
ϕ : volume fraction of the spherical particles of radius a

REFERENCES

Batchelor, G. K., "The Stress System in a Suspension of Force-free Particles", *J. Fluid Mech.,* **41**, 545 (1970).

Batchelor, G. K., "Transport Properties of Two-phase Materials with Random Structure", *Ann. Rev. Fluid Mech.,* **6**, 227 (1974).

Batchelor, G. K. and Green, J. T., "The Hydrodynamic Interaction of Two Small Freely-moving Spheres in a Linear Flow Field", *J. Fluid Mech.,* **56**, 375 (1972a).

Batchelor, G. K. and Green, J. T., "The Determination of the Bulk Stress in a Suspension of Spherical Particles to Order c^2", *J. Fluid Mech.,* **56**, 401 (1972b).

Fowler, M. E., Keskkula, H. and Paul, D. R., "Synergistic Toughening in Rubber Modified Blends", *Polymer,* **28**, 17 (1987).

Gebizlioglu, O. S., Beckham, H. W., Argon, A. S. and Cohen, R. E., "A New Mechanism of Toughening Glassy Polymers. 1. Experimental Procedures", *Macromolecules,* **23**, 3968 (1990).

Laurienzo, P., Malinconico, M., Martuscelli, E., Ragosta, G. and Volpe, M. G., "Synthesis and Structure-property Relationships of a New Class of Rubber-toughened PMMA", *J. Applied Polymer Science,* **44**, 1883 (1992).

Leal, L. G., "Laminar Flow and Convective Transport Processes : Scaling Principles and Asymptotic Analysis", Butterworth-Heinemann, Boston, 1992.

Lovell, P. A., McDonald, J., Saunders, D. E. J., Sherratt, M. N. and Young, R. J., "Mechanical Properties of Rubber-toughened Poly (methyl methacrylate)", *Plastics, Rubber and Composites Processing and Applications,* **16**, 37 (1991).

Matinis, V. A., "The Interfacial Stresses in Particulate Composite Systems", *Polymer Engineering and Science,* **9**, 100 (1969).

Moshev, V. V., "Micromechanical Features of Highly-filled Composites are Most Distinctly Defined by Mechanical Heterogeneity", *Int. J. Polymeric. Mater.,* **8**, 153 (1980).

Ricco, T., Pavan, A. and Danusso, F., "Micromechanical Analysis of a Model for Particulate Composite Materials with Composite Particles-Survey of Craze Initiation", *Polymer Engineering and Science,* **18**, 774 (1978).

Wang, T. T., Matsuo, M. and Kwei, T. K., "Criteria of Craze Initiation in Glassy Polymers", *J. Applied Physics,* **42**, 4188 (1971).